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TEXTBOOK TESTBANKS

Solutions Manual for A History of Mathematics 4th Katz

SOLUTIONS MANUAL SAMPLE PREVIEW

PUBLISHER

Pearson

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RESOURCE TYPE

Solutions Manual

ISBN

9780136839118

CHAPTER 1

1. The answers are given in the answer section of the text. For the Egyptian hieroglyphics, 375 is three hundreds, seven tens, and five ones, while 4856 is four thousands, eight hundreds, five tens, and six ones. For Babylonian cuneiform, note that $375 = 6 \times 60 + 15$, while $4856 = 1 \times 3600 + 20 \times 60 + 56$.

2.

1	34
$'2$	68
4	136
8	272
$\underline{'16}$	$\underline{544}$
18	612

1	5	
10	$50'$	(multiply by 10)
2	10	(double first line)
4	20	(double third line)
8	$40'$	(double fourth line)
$\overline{2}$	$2\,\overline{2}'$	(halve first line)
$\underline{\overline{10}}$	$\underline{\overline{2}}'$	(invert third line)
$18\,\overline{2}\,\overline{10}$	93	

3.

1	$\overline{2}\ \overline{14}$
$\overline{2}$	$\overline{4}\ \overline{28}$
$\overline{4}$	$\underline{\overline{8}\ \overline{56}}$
	1

4.

1	$\overline{28}$
$\overline{2}$	$\overline{56}$
$\overline{4}$	$\underline{\overline{112}}$
	$\overline{16}$

5. We multiply 10 by $\overline{\overline{3}}\ \overline{30}$:

1		$\overline{\overline{3}}$	$\overline{30}$	
′2	1	$\overline{3}$	$\overline{15}$	
4	2	$\overline{\overline{3}}$	$\overline{10}$	$\overline{30}$
′8	5	$\overline{2}$	$\overline{10}$	

The total of the two marked lines is then 7, as desired.

6.

1	7	$\overline{2}$	$\overline{4}$	$\overline{8}$
2	15	$\overline{2}$	$\overline{4}$	
4	31	$\overline{2}$		
8	63			
$\overline{\overline{3}}$	5	$\overline{4}$		

Note that the sum of the three last terms in the second column is 99 $\overline{2}\ \overline{4}$. We therefore need to figure out by what to multiply 7 $\overline{2}\ \overline{4}\ \overline{8}$ to give $\overline{4}$ so that we get a total of 100. But since we know from the fourth line that multiplying that value by 8 gives 63, we also know that multiplying it by $\overline{63}$ gives $\overline{8}$. Thus the required number is double $\overline{63}$, which is $\overline{42}\ \overline{126}$. Thus the final result of our division is 12 $\overline{\overline{3}}\ \overline{42}\ \overline{126}$.

7.

1	$7\ \overline{2}\ \overline{4}\ \overline{8}$
2	$15\ \overline{2}\ \overline{4}$
$'4$	$31\ \overline{2}$
$'8$	63
$'\overline{\overline{3}}$	$4\ \overline{\overline{3}}\ \overline{3}\ \overline{6}\ \overline{12}$
$12\ \overline{\overline{3}}$	$98\ \overline{2}\ \overline{\overline{3}}\ \overline{3}\ \overline{6}\ \overline{12}$ $99\ \overline{2}\ \overline{4}$

8.

$2 \div 11$	1	11	$2 \div 23$	1	23
	$\overline{\overline{3}}$	$7\ \overline{3}$		$\overline{\overline{3}}$	$15\ \overline{3}$
	$\overline{3}$	$3\ \overline{\overline{3}}$		$\overline{3}$	$7\ \overline{\overline{3}}$
	$\overline{6}$	$1\ \overline{\overline{3}}\ \overline{6}'$		$\overline{6}$	$3\ \overline{2}\ \overline{3}$
	$\overline{66}$	$\overline{6}'$		$\overline{12}$	$1\ \overline{2}\ \overline{4}\ \overline{6}'$
				$\overline{276}$	$\overline{12}'$
	$\overline{6}\ \overline{66}$	2		$\overline{12}\ \overline{276}$	2

9. $x + \frac{1}{7}x = 19$. Choose $x = 7$; then $7 + \frac{1}{7} \cdot 7 = 8$. Since $19 \div 8 = 2\frac{3}{8}$, the correct answer is $2\frac{3}{8} \times 7 = 16\frac{5}{8}$.

10. $(x + \frac{2}{3}x) - \frac{1}{3}(x + \frac{2}{3}x) = 10$. In this case, the "obvious" choice for x is $x = 9$. Then 9 added to 2/3 of itself is 15, while 1/3 of 15 is 5. When you subtract 5 from 15, you get 10. So in this case our "guess" is correct.

11. The equation here is $(1 + \frac{1}{3} + \frac{1}{4})x = 2$. Therefore. we can find the solution by dividing 2 by $1 + \frac{1}{3} + \frac{1}{4}$. We set up that problem:

1	$1\ \overline{2}\ \overline{4}$
$\overline{\overline{3}}$	$1\ \overline{18}$
$\overline{3}$	$\overline{2}\ \overline{36}$
$\overline{6}$	$\overline{4}\ \overline{72}$
$\overline{12}$	$\overline{8}\ \overline{144}$

The sum of the numbers in the right-hand column beneath the initial line is $1\frac{141}{144}$. So we need to find multipliers giving us $\frac{3}{144} = \overline{144}\ \overline{72}$. But $1\ \overline{3}\ \overline{4}$ times 144 is 228. It follows that multiplying $1\ \overline{3}\ \overline{4}$ by $\overline{228}$ gives $\overline{144}$ and multiplying by $\overline{114}$ gives $\overline{72}$. Thus, the answer is $1\ \overline{6}\ \overline{12}\ \overline{114}\ \overline{228}$.

12. Since there are 10 hekats to be divided among 10 men, the average for each is 1 hekat. So to get the largest share, we should add to the average 1/8 times half the number of differences. Since there are 9 differences, we add 1/8 $4\frac{1}{2}$ times, or 9/16. Therefore, the largest share is $1\frac{9}{16}$ hekats, which the scribe writes as $1\ \overline{2}\ \overline{16}$. We then subtract 1/8 from this value 9 times to get the share of each man. The answers are $1\frac{9}{16}$, $1\frac{7}{16}$, $1\frac{5}{16}$, $1\frac{3}{16}$, $1\frac{1}{16}$, $\frac{15}{16}$, $\frac{13}{16}$, $\frac{11}{16}$, $\frac{9}{16}$, and $\frac{7}{16}$.

13. Since x must satisfy $100 : 10 = x : 45$, we would get that $x = \frac{45\times 100}{10}$; the scribe breaks this up into a sum of two parts, $\frac{35\times 100}{10}$ and $\frac{10\times 100}{10}$.

14. The ratio of the cross section area of a log of 5 handbreadths in diameter to one of 4 handbreadths diameter is $5^2 : 4^2 = 25 : 16 = 1\frac{9}{16}$. Thus, 100 logs of 5 handbreadths diameter are equivalent to $1\frac{9}{16} \times 100 = 156\frac{1}{4}$ logs of 4 handbreadths diameter.

16. The modern formula for the surface area of a half-cylinder of diameter d and height h is $A = \frac{1}{2}\pi dh$. Similarly, the modern formula for the surface area of a hemisphere of diameter d is $A = \frac{1}{2}\pi d^2$. These formulas are identical if $h = d$.

17. 7/5 = 1;24 13/15 = 0;52 11/24 = 0;27,30 33/50 = 0;39,36

18. 0;22,30 = 3/8 0;08,06 = 27/200 0;04,10 = 5/72 0;05,33,20 = 5/54

19. Since 60 is $(2\frac{1}{2}) \times 24$, the reciprocal of 24 is 2;30. Since 60 is $(1\frac{7}{8}) \times 32$, and $\frac{7}{8}$ can be expressed as 0;52,30 the reciprocal of 32 is 1;52,30. Since 60 is $(1\frac{1}{3}) \times 45$, the reciprocal of 45 is 1:20. Since $60 = 1\frac{1}{9} \times 54$, and $\frac{1}{9}$ can be expressed as $\frac{1}{10} + \frac{1}{90} = \frac{6}{60} + \frac{40}{3600} =$ 0;06,40, the reciprocal of 54 is 1;06,40. If the only prime divisors of n are 2, 3, 5, then n is a regular sexagesimal.

20. 25 × 1,04 = 1,40 + 25,00 = 26,40. 18 × 1,21 = 6,18 + 18,00 = 24,18. 50 ÷ 18 = 50 × 0;3,20 = 2;30 + 0;16,40 = 2;46,40. 1,21 ÷ 32 = 1,21 × 0;01,52,30 = 1;21 + 1;10,12 + 0;00,40,30 = 2;31,52,30.

21. 1,08,16 × 3,45 = 4,16,0,0. 4,16 × 3,45 = 16,0,0. 16 × 3,45 = 1,0,0.

22. Since the length of the circumference C is given by $C = 4a$, and because $C = 6r$, it follows that $r = \frac{2}{3}a$. The length T of the long transversal is then $T = r\sqrt{2} = (\frac{2}{3}a)(\frac{17}{12}) = \frac{17}{18}a$. The length t of the short transversal is $t = 2(r - \frac{t}{2}) = 2a(\frac{2}{3} - \frac{17}{36}) = \frac{7}{18}a$. The area A of the barge is twice the difference between the area of a quarter circle and the area of the right triangle formed by the long transversal and two perpendicular radii drawn from the two ends of that line. Thus

$$A = 2\left(\frac{C^2}{48} - \frac{r^2}{2}\right) = 2\left(\frac{a^2}{3} - \frac{2a^2}{9}\right) = \frac{2}{9}a^2.$$

23. Since the length of the circumference C is given by $C = 3a$, and because $C = 6r$, it follows that $r = \frac{a}{2}$. The length T of the long transversal is then $T = r\sqrt{3} = (\frac{a}{2})(\frac{7}{4}) = \frac{7}{8}a$. The length t of the short transversal is twice the distance from the midpoint of the arc to the center of the long transversal. If we set up our circle so that it is centered on the origin, the midpoint of the arc has coordinates $(\frac{r}{2}, \frac{\sqrt{3}r}{2})$ while the midpoint of the long transversal has coordinates $(\frac{r}{4}, \frac{\sqrt{3}r}{4})$. Thus the length of half of the short transversal is $\frac{r}{2}$ and then $t = r = \frac{a}{2}$. The area A of the bull's eye is twice the difference between the area of a third of a circle and the area of the triangle formed by the long transversal and radii drawn from the two ends of that line. Thus

$$A = 2\left(\frac{C^2}{36} - \frac{1}{2}\frac{r}{2}T\right) = 2\left(\frac{9a^2}{36} - \frac{1}{2}\frac{a}{4}\frac{7a}{8}\right) = 2a^2\left(\frac{1}{4} - \frac{7}{64}\right) = \frac{9}{32}a^2.$$

24. If a is the length of one of the quarter-circle arcs defining the concave square, then the diagonal is equal to the diameter of that circle. Since the circumference is equal to $4a$, the diameter is one-third of that circumference, or $1\frac{1}{3}a$. The transversal is equal to the diagonal of the circumscribing square less the diameter of the circle (which is equal to the side of the square). Since the diagonal of a square is approximated by 17/12 of the side, the transversal is therefore equal to 5/12 of the diameter, or $\frac{5}{12}\frac{4}{3}a = \frac{5}{9}a$.

25. $\sqrt{3} = \sqrt{2^2 - 1} \approx 2 - \frac{1}{2}\cdot 1 \cdot \frac{1}{2} = 2 - 0;15 = 1;45$. Since an approximate reciprocal of 1;45 is 0;34,17,09, we get further that $\sqrt{3} = \sqrt{(1;45)^2 - 0;03,45} = 1;45 - (0;30)(0;03,45)(0;34,17,09) = 1;45 - 0;01,04,17,09 = 1;43,55,42,51$, which we truncate to 1;43,55,42 because we know this value is a slight over-approximation.

26. $v + u = 1;48 = 1\frac{4}{5}$ and $v - u = 0;33,20 = \frac{5}{9}$. So $2v = 2;21,20$ and $v = 1;10,40 = \frac{106}{90}$. Similarly, $2u = 1;14,40$ and $u = 0;37,20 = \frac{56}{90}$. Multiplying by 90 gives $x = 56$, $d = 106$. In the second part, $v + u = 2;05 = 2\frac{1}{12}$ and $v - u = 0;28,48 = \frac{12}{25}$. So $2v = 2;33,48$ and $v = 1;16,54 = \frac{769}{600}$. Similarly, $2u = 1;36,12$ and $u = 0;48,06 = \frac{481}{600}$. Multiplying by 600 gives $x = 481$, $d = 769$. Next, if $v = \frac{481}{360}$ and $u = \frac{319}{360}$, then $v + u = 2\frac{2}{9} = 2;13,20$. Finally, if $v = \frac{289}{240}$ and $u = \frac{161}{240}$, then $v + u = 1\frac{7}{8} = 1;52,30$.

27. The equations for u and v can be solved to give $v = 1:22,08,27 = \frac{295707}{216000} = \frac{98569}{72000}$ and $u = 0;56,05,57 = \frac{201957}{216000} = \frac{67319}{72000}$. Thus the associated Pythagorean triple is 67319, 72000, 98569.

28. The two equations are $x^2 + y^2 = 1525$; $y = \frac{2}{3}x + 5$. If we substitute the second equation into the first and simplify, we get $13x^2 + 60x = 13500$. The solution is then $x = 30$, $y = 25$.

29. If we guess that the length of the rectangle is 60, then the width is 45 and the diagonal is $\sqrt{60^2+45^2} = 75$. Since this value is $1\frac{7}{8}$ times the given value of 40, the correct length of the rectangle should be $60 \div 1\frac{7}{8} = 32$. Then the width is 24.

30. One way to solve this is to let x and $x-600$ be the areas of the two fields. Then the equation is $\frac{2}{3}x+\frac{1}{2}(x-600) = 1100$. This reduces to $\frac{7}{6}x = 1400$, so $x = 1200$. The second field then has area 600.

31. Let x be the weight of the stone. The equation to solve is then $x-\frac{1}{7}x-\frac{1}{13}(x-\frac{1}{7}x) = 60$. We do this using false position twice. First, set $y = x - \frac{1}{7}x$. The equation in y is then $y - \frac{1}{13}y = 60$. We guess $y = 13$. Since $13 - \frac{1}{13}13 = 12$, instead of 60, we multiply our guess by 5 to get $y = 65$. We then solve $x - \frac{1}{7}x = 65$. Here we guess $x = 7$ and calculate the value of the left side as 6. To get 65, we need to multiply our guess by $\frac{65}{6} = 10\frac{1}{6}$. So our answer is $x = 7 \times \frac{65}{6} = 75\frac{5}{6}$ *gin*, or 1 *mina* $15\frac{5}{6}$ *gin*.

32. We do this in three steps, each using false position. First, set $z = x - \frac{1}{7}x + \frac{1}{11}(x - \frac{1}{7}x)$. The equation for z is then $z - \frac{1}{13}z = 60$. We guess 13 for z and calculate the value of the left side to be 12, instead of 60. Thus we must multiply our original guess by 5 and put $z = 65$. Then set $y = x - \frac{1}{7}x$. The equation for y is $y+\frac{1}{11}y = 65$. If we now guess $y = 11$, the result on the left side is 12, instead of 65. So we must multiply our guess by $\frac{65}{12}$ to get $y = \frac{715}{12} = 59\frac{7}{12}$. We now solve $x - \frac{1}{7}x = 59\frac{7}{12}$. If we guess $x = 7$, the left side becomes 6 instead of $59\frac{7}{12}$. So to get the correct value, we must multiply 7 by $\frac{715}{12}/6 = \frac{715}{72}$. Therefore, $x = 7 \times \frac{715}{72} = \frac{5005}{72} = 69\frac{37}{72}$ *gin* = 1 *mina* $9\frac{37}{72}$ *gin*.

33. Start with a square of side x and cut off a strip of width a from the right side. The remaining rectangle then has area $x^2 - ax$, or b. This rectangle can then be thought of as a square of side $x-a/2$ that is missing a small square of side $a/2$. If one adds back that small square, then the square of side $x-a/2$ has area $b+(a/2)^2$, so we can find x.

34. The equation $x - \frac{60}{x} = 7$ is equivalent to $x^2 - 60 = 7x$ or to $x^2 - 7x = 60$. The solution is then $x = \sqrt{(\frac{7}{2})^2+60}+\frac{7}{2} = \frac{17}{2}+\frac{7}{2} = 12$. Thus the two numbers are 12 and 5.

35. Given the appropriate coefficients, the equation becomes $\frac{4}{9}a^2+a+\frac{4}{3}a = \frac{23}{18}$, where a is the length of the arc. If we scale up by $\frac{4}{9}$, we get the equation $(\frac{4}{9}a)^2+\frac{7}{3}(\frac{4}{9}a) = \frac{46}{81}$. The algorithm for this type of equation gives $\frac{4}{9}a = \sqrt{(\frac{7}{6})^2+\frac{46}{81}} - \frac{7}{6} = \frac{25}{18} - \frac{7}{6} = \frac{2}{9}$. Thus $a = \frac{1}{2}$.

36. The equation is $\frac{2}{3}x^2+\frac{1}{3}x = \frac{1}{3}$. To solve, we scale by $\frac{2}{3}$: $(\frac{2}{3}x)^2+\frac{1}{3}(\frac{2}{3}x) = \frac{2}{9}$. The solution is $\frac{2}{3}x = \sqrt{(\frac{1}{6})^2+\frac{2}{9}} - \frac{1}{6} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. Thus $x = \frac{1}{2}$.

37. All the triangles in Figure 1.15 are similar to one another, and therefore their sides are all in the ratio 3:4:5. Therefore, $AE = \frac{3}{5}AD = 0;36 \times 0;36 = 0;21,36$. Also, $DE = \frac{4}{5}AD = 0;48 \times 0;36 = 0;28,48$. Also, $EF = \frac{4}{5}DE = 0;48 \times 0;28,48 = 0;23,02,24$. Finally, $DF = \frac{3}{5}DE = 0;36 \times 0;28,48 = 0;17,16,48$.

38. If the circumference is 60, then the radius is 10. Thus, if the distance of the chord from the circumference is x, then we have a right triangle of sides 6 and $10-x$, with hypotenuse 10. The Pythagorean theorem leads to the equation $6^2+(10-x)^2 = 100$, or $x^2+36 = 20x$, for which the only valid solution is $x = 2$.

39. The two equations are $\ell + w = 7$, $\ell w + \frac{1}{2}\ell + \frac{1}{3}w = 15$. To put this into a standard Babylonian form, we can rewrite the second equation in the form $(\ell + \frac{1}{3})(w + \frac{1}{2}) = 15\frac{1}{6}$. We can then rewrite the first equation as $(\ell + \frac{1}{3}) + (w + \frac{1}{2}) = 7\frac{5}{6}$. Then the Babylonian algorithm yields $\ell + \frac{1}{3} = \frac{47}{12} + \sqrt{(\frac{47}{12})^2 - \frac{91}{6}} = \frac{47}{12} + \frac{5}{12} = \frac{13}{3}$. Therefore, $\ell = 4$ and so $w = 3$.

CHAPTER 2

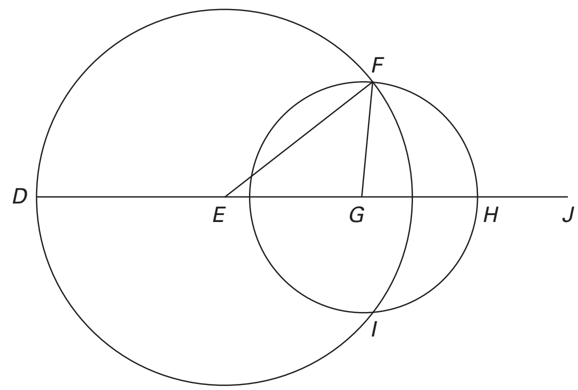
1. $125 = \rho\kappa\epsilon$, $62 = \xi\beta$, $4821 = {}'\delta\omega\kappa\alpha$, $23,855 = M^{\beta}\,{}'\gamma\omega\nu\epsilon$
2. $\frac{8}{9} = \angle\ \acute{\gamma}\ \acute{\iota}\acute{\eta}$ $(8/9 = 1/2 + 1/3 + 1/18)$
3. The answer is in the back of the text. The basic idea is that $200/9 = 22\frac{2}{9} = 22 + 1/6 + 1/18$.
4. The average of a and c is $1/4 + 1/16 + 1/64$. The average of b and d is $1/2 + 1/4 + 1/8 + 1/16$. The product of the two averages is $1/8 + 1/16 + 1/32 + 1/64 + 1/32 + 1/64 + 1/128 + 1/256 + 1/128 + 1/256 + 1/512 + 1/1024$, or $1/4 + 59/1024$. This is slightly less than the given answer of $1/4 + 1/16$.
5. Since $AB = BC$; since the two angles at B are equal; and since the angles at A and C are both right angles, it follows by the angle-side-angle theorem that $\triangle EBC$ is congruent to $\triangle SBA$ and therefore that $SA = EC$.
6. Because both angles at E are right angles; because AE is common to the two triangles; and because the two angles CAE are equal to one another, it follows by the angle-side-angle theorem that $\triangle AET$ is congruent to $\triangle AES$. Therefore $SE = ET$.
7. The distance from the center of the pyramid to the tip of the shadow is $378 + 342 = 720$ feet. Therefore the height of the pyramid is $6/9 = 2/3$ of this value, or 480 feet.
8. $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Therefore the oblong number $n(n+1)$ is double the triangular number T_n.
9. $n^2 = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$, and the summands are the triangular numbers T_{n-1} and T_n.
10. $\frac{8n(n+1)}{2} + 1 = 4n^2 + 4n + 1 = (2n+1)^2$. $(2n+1)^2 - 1 = 4n^2 + 4n = \frac{8n(n+1)}{2}$.
11. Suppose $a^2 + b^2 = c^2$. Suppose a is odd. Then a^2 is odd. If b is odd, then b^2 is odd and c^2 is even, so c is even. If b is even, then b^2 is even and c^2 is odd, so c is odd. A similar result holds if c is odd.
12. Examples using the first formula are (3,4,5), (5,12,13), (7,24,25), (9,40,41), (11,60,61). Examples using the second formula are (8,15,17), (12,35,37), (16,63,65), (20,99,101), (24,143,145).
13. Let us assume that the second leg is commensurable to the first and let b, a be numbers representing the two legs (in terms of some unit). We may as well assume that b and a are relatively prime. Since the hypotenuse is double the first leg, we have $b^2 + a^2 = (2a)^2 = 4a^2$, or $b^2 = 3a^2$. Since b^2 is a multiple of 3, it must also be a multiple of 9, so $b^2 = 9c^2$ and $b = 3c$. Then $9c^2 = 3a^2$, or $a^2 = 3c^2$. This implies that a^2 is a multiple of 9, so that a is a multiple of 3. But then both a and b are multiples of 3, contradicting the fact that they are relatively prime.

14. Since similar segments are to their corresponding circles in the same ratio, the areas of similar segments are to one another as the squares on the diameters of the circles. Thus, the areas of similar segments are also to one another as the squares on the radii of the circles. But in similar segments, the triangles formed by the two radii and chords are similar triangles. Thus the chord of one segment is to the chord in the similar segment as the radius of the first circle to the radius of the second. That is, the squares on the radii are to one another as the squares on the chords. Therefore, the areas of similar segments are to one another as the squares on their chords.
15. By Exercise 14, the area of segment *BD* is the area of segment *AB* as the square on *BD* is the square on *AB*. But this ratio is equal to 3. Thus, the area of segment *BD* is three times the area of segment *AB*, or is equal to the sum of the areas of segments *AB*, *AC*, and *CD*. Therefore, the area of lune is equal to the difference between the area of the large segment and the area of segment *BD*. But this is equal to the difference between the area of the large segment and the areas of the three small segments, which is in turn equal to the area of the trapezoid. To construct the trapezoid, note that one can certainly construct a line segment equal to $\sqrt{3}$ times the length of a given line segment. To place this line segment both parallel to the original one and such that the lines connecting the endpoints of the two segments are each equal to the original line segment, we simply need to find the distance between the two segments. And that can be constructed by using the Pythagorean Theorem applied to the triangle whose hypotenuse is equal to the original segment and one leg of which is equal to half the difference between the new line segment and the original one. To circumscribe a circle around this trapezoid, note that one can construct a circle through three points, say *B*, *A*, and *C*. By the symmetry of the trapezoid, this circle will also go through point *D*.
21. If one equates the times of the two runners, where d is the distance traveled by Achilles, the equation is $d/10 = (d - 500)/(1/5)$. This is equivalent to $49d = 25{,}000$, so $d =$ 510.2 yards. Since Achilles is traveling at 10 yards per second, this will take him 51.02 seconds.

CHAPTER 3

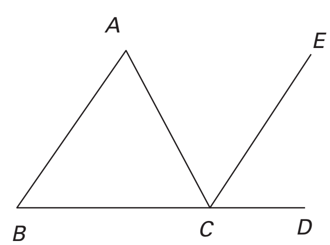
1. One way to do this is to use I-4. Namely, consider the isosceles triangle *ABC* with equal sides *AB* and *BC* also as a triangle *CBA*. Then the triangles *ABC* and *CBA* have two sides equal to two sides and the included angles also equal. Thus, by I-4, they are congruent. Therefore, angle *BAC* is equal to angle *BCA*, and the theorem is proved.
2. Put the point of the compass on the vertex *V* of the angle and swing equal arcs intersecting the two legs at *A* and *B*. Then place the compass at *A* and *B*, respectively, and swing equal arcs, intersecting at *C*. The line segment connecting *V* to *C* then bisects the angle. To show that this is correct, note that triangles *VAC* and *VBC* are congruent by *SSS*. Therefore, the two angles *AVC* and *BVC* are equal.
3. Let the lines *AB* and *CD* intersect at *E*. Then angles *AEB* and *CED* are both straight angles, angles equal to two right angles. If one subtracts the common angle *CEB* from each of these, the remaining angles *AEC* and *BED* are equal, and these are the vertical angles of the theorem.

4. Suppose the three lines have length a, b, and c, with $a \geq b \geq c$. On the straight line DH of length $a+b+c$, let the length of DF be a, the length of FG be b, and the length of GH be c. Then draw a circle centered on F with radius a and another circle centered at G with radius c. Let K be an intersection point of the two circles. Then connect FK and GK. Triangle FKG will then be the desired triangle. For $FK = FD$, and this has length a. Also FG has length b, while $GK = GH$ and this has length c. Note also that we must have $b+c > a$, for otherwise the two circles would not intersect. That $a+b > c$ and $a+c > b$ are obvious from how we have labeled the three lengths.



5. Suppose angle DCE is given, and we want to construct an angle equal to angle DCE at point A of line AB. Draw the line DE so that we now have a triangle DCE. (Here D and E are arbitrary points along the two arms of the given angle.) Then, by the result of Exercise 4, construct a triangle AGF, with AG along line AB, where $AG = CE$, $AF = CD$, and $FG = DE$. By the side-side-side congruence theorem, triangle AGF is congruent to triangle CED. Therefore, angle FAG is equal to angle DCE, as desired.

6. Let ABC be the given triangle. Extend BC to D and draw CE parallel to AB. By I–29, angles BAC and ACE are equal, as are angles ABC and ECD. Therefore angle ACD equals the sum of the angles ABC and BAC. If we add angle ACB to each of these, we get that the sum of the three interior angles of the triangle is equal to the straight angle BCD. Because this latter angle equals two right angles, the theorem is proved.

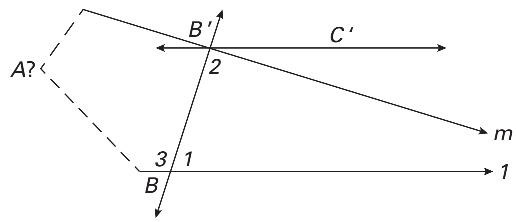


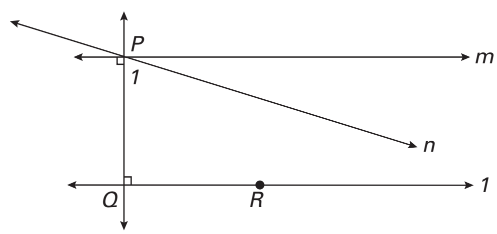
7. Place the given rectangle $BEFG$ so that BE is in a straight line with AB. Extend FG to H so that AH is parallel to BG. Connect HB and extend it until it meets the extension of

FE at D. Through D draw DL parallel to FH and extend GB and HA so they meet DL in M and L, respectively. Then HD is the diagonal of the rectangle $FDLH$ and so divides it into two equal triangles HFD and HLD. Because triangle BED is equal to triangle BMD and also triangle BGH is equal to triangle BAH, it follows that the remainders, namely rectangles $BEFG$ and $ABML$, are equal. Thus $ABML$ has been applied to AB and is equal to the given rectangle $BEFG$.

8. Because triangles ABN, ABC, and ANC are similar, we have $BN : AB = AB : BC$, so $AB^2 = BN \cdot BC$, and $NC : AC = AC : BC$, so $AC^2 = NC \cdot BC$. Therefore $AB^2 + AC^2 = BN \cdot BC + NC \cdot BC = (BN + NC) \cdot BC = BC^2$, and the theorem is proved.

9. In this proof, we shall refer to certain propositions in Euclid's Book I, all of which are proved before Euclid first uses postulate 5. (That occurs in proposition 29.) First, assume Playfair's Axiom. Suppose line t crosses lines m and l and that the sum of the two interior angles (angles 1 and 2 in the diagram) is less than two right angles. We know that the sum of angles 1 and 3 is equal to two right angles. Therefore $\angle 2 < \angle 3$. Now on line BB' and point B' construct line $B'C'$ such that $\angle C'B'B = \angle 3$ (Proposition 23). Therefore, line $B'C'$ is parallel to line l (Proposition 27). Therefore, by Playfair's Axiom, line m is not parallel to line l. It therefore meets l. We must show that the two lines meet on the same side as C'. If the meeting point A is on the opposite side, then $\angle 2$ is an exterior angle to triangle ABB', yet it is smaller than $\angle 3$, one of the interior angles, contradicting Proposition 16. We have therefore derived Euclid's postulate 5.

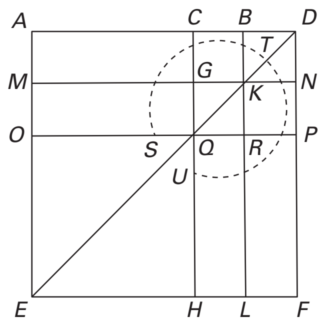




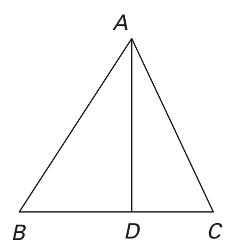
Second, assume Euclid's postulate 5. Let l be a given line and P a point outside the line. Construct the line t perpendicular to l through P (Proposition 12). Next, construct the line m perpendicular to line t at P (Proposition 11). Since the alternate interior angles formed by line t crossing lines m and l are both right and therefore are equal, it follows from Proposition 27 that m is parallel to l. Now suppose n is any other line through P. We will show that n meets l and is therefore not parallel to l. Let $\angle 1$ be the acute angle that n makes with t. Then the sum of angle 1 and angle PQR is less than two right angles. By postulate 5, the lines meet.

Note that in this proof, we have actually proved the equivalence of Euclid's postulate 5 to the statement that given a line l and a point P not on l, there is at most one line through P which is parallel to l. The other part of Playfair's Axiom was proved (in the second part above) without use of postulate 5 and was not used at all in the first part.

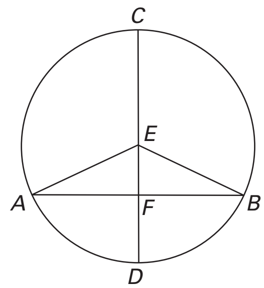
10. One possibility for an algebraic translation: If the line has length a and is cut at a point with coordinate x, then $4ax + (a - x)^2 = (a + x)^2$. This is a valid identity. Here is a geometric diagram, with $AB = ME = a$ and $CB = BD = BK = KR = x$:



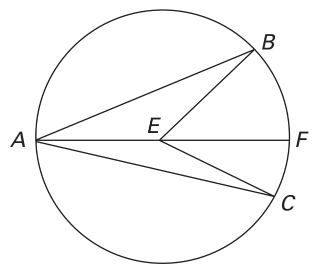
11. If ABC is the given acute-angled triangle and AD is perpendicular to BC, then the theorem states that the square on AC is less than the squares on CB and BA by twice the rectangle contained by CB and BD. If we label AC as b, BA as c, and CB as a, then $BD = c\cos B$. Thus the theorem can be translated algebraically into the form $b^2 = a^2 + c^2 - 2ac\cos B$, exactly the law of cosines in this case.



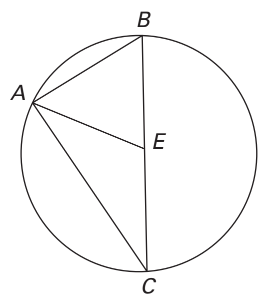
12. Suppose the diameter CD of a circle with center E bisects the chord AB at F. Then join EA and EB, forming triangle EAB. Triangles AEF and BEF are congruent by side-side-side (since $AE = BE$ are both radii of the circle and F bisects AB). Therefore angles EFA and EFB are equal. But the sum of those two angles is equal to two right angles. Hence each is a right angle, as desired. To prove the converse, use the same construction and note that triangle AEB is isosceles, so angle EAF is equal to angle EBF, while both angles EFA and EFB are right by hypothesis. It follows that triangles AEF and BEF are again congruent, this time by angle-angle-side. So $AF = BF$, and the diameter bisects the chord.



13. In the circle ABC, let the angle BEC be an angle at the center and the angle BAC be an angle at the circumference which cuts off the same arc BC. Connect $\angle EAB$. Similarly, $\angle FEC$ is double $\angle EAC$. Therefore the entire $\angle BEC$ is double the entire $\angle BAC$. Note that this argument holds as long as line EF is within $\angle BEC$. If it is not, an analogous argument by subtraction holds.

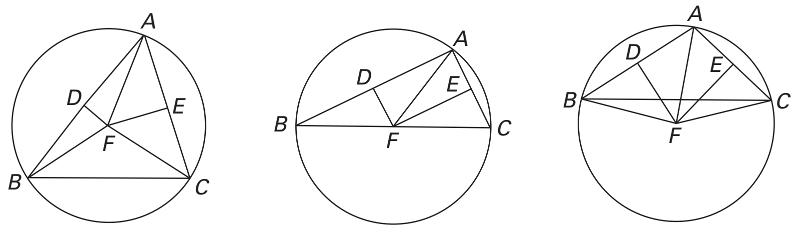


14. Let $\angle BAC$ be an angle cutting off the diameter BC of the circle. Connect A to the center E of the circle. Since $EB = EA$, it follows that $\angle EBA = \angle EAB$. Similarly, $\angle ECA = \angle EAC$. Therefore the sum of $\angle EBA$ and $\angle ECA$ is equal to $\angle BAC$. But the sum of all three angles equals two right angles. Therefore, twice $\angle BAC$ is equal to two right angles, and angle BAC is itself a right angle.

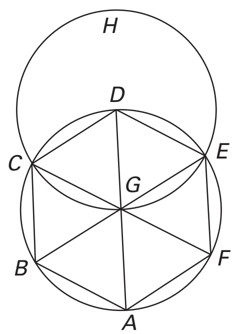


15. Let triangle ABC be given. Let D be the midpoint of AB and E the midpoint of AC. Draw a perpendicular at D to AB and a perpendicular at E to AC and let them meet at point F (which may be inside or outside the triangle, or on side BC). Assume first that

F is inside the triangle, and connect FB, FA, and FC. Since $BD = BA$, triangles FDB and FDA are congruent by side-angle-side. Therefore $FB = FA$. Similarly, triangles FEA and FEC are congruent. So $FC = FA$. Therefore all three lines FA, FB, and FC are equal, and a circle can be drawn with center F and radius equal to FA. This circle will circumscribe the given triangle. Finally, note that the identical construction works if F is on line BC or if F is outside the triangle.



16. Let G be the center of the given circle and AGD a diameter. With center at D and radius DG, construct another circle. Let C and E be the two intersections of the two (equal) circles, and connect DC and DE. Then DE and DC are two sides of the desired regular hexagon. To find the other four sides, draw the diameters CGF and EGB. Then CB, BA, AF, and FE are the other sides. To demonstrate that we have in fact constructed a regular hexagon, note that all the triangles whose bases are sides of the hexagon and whose other sides are radii are equilateral; thus all the sides of the hexagon are equal and all the angles of the hexagon are also equal.



17. In the circle, inscribe a side AC of an equilateral triangle and a side AB of an equilateral pentagon. Then arc BC is the difference between one-third and one-fifth of the circumference of the circle. That is, arc $BC = \frac{2}{15}$ of the circumference. Thus, if we bisect that arc at E, then lines BE and EC will each be a side of a regular 15-gon.

18. Let $a = s_1 b + r_1$, $b = s_2 r_1 + r_2$, . . ., $r_{k-1} = s_{k+1} r_k$. Then r_k divides r_{k-1} and therefore also $r_{k-2}, \ldots, b, a$. If there were a greater common divisor of a and b, it would divide $r_1, r_2, \ldots, r_k$. Since it is impossible for a greater number to divide a smaller, we have shown that r_k is in fact the greatest common divisor of a and b.

19.

$$\begin{aligned}963 &= 1\cdot 657+306\\ 657 &= 2\cdot 306+45\\ 306 &= 6\cdot 45+36\\ 45 &= 1\cdot 36+9\\ 36 &= 4\cdot 9+0\end{aligned}$$

Therefore, the greatest common divisor of 963 and 657 is 9.

$$\begin{aligned}4001 &= 1\cdot 2689+1312\\ 2689 &= 2\cdot 1312+65\\ 1312 &= 20\cdot 65+12\\ 65 &= 5\cdot 12+5\\ 12 &= 2\cdot 5+2\\ 5 &= 2\cdot 2+1\end{aligned}$$

Therefore, the greatest common divisor of 4001 and 2689 is 1.

20.

$$\begin{aligned}46 &= 7\cdot 6+4 & 23 &= 7\cdot 3+2\\ 6 &= 1\cdot 4+2 & 3 &= 1\cdot 2+1\\ 4 &= 2\cdot 2 & 2 &= 2\cdot 1\end{aligned}$$

Note that the multiples 7, 1, 2 in the first example equal the multiples 7, 1, 2 in the second.

21.

$$\begin{aligned}33 &= 2\cdot 12+9 & 11 &= 2\cdot 4+3\\ 12 &= 1\cdot 9+3 & 4 &= 1\cdot 3+1\\ 9 &= 3\cdot 3 & 3 &= 3\cdot 1\end{aligned}$$

It follows that both ratios can be represented by the sequence $(2, 1, 3)$.

22. Since $1 - x = x^2$, we have

$$1 = 1\cdot x + (1-x) = 1\cdot x + x^2$$

$$x = 1\cdot x^2 + (x - x^2) = 1\cdot x^2 + x(1-x) = 1\cdot x^2 + x^3$$

$$x^2 = 1\cdot x^3 + (x^2 - x^3) = 1\cdot x^3 + x^2(1-x) = 1\cdot x^3 + x^4$$

$$\ldots$$

Thus $1 : x$ can be expressed in the form $(1, 1, 1, \ldots)$.

23. If d is the diagonal of a square of side s, then the first division gives $d = 1s + r$. To understand the next steps, it is probably easiest to set $s = 1$ and deal with the numerical values. Therefore, $d = \sqrt{2}$, and $r = \sqrt{2} - 1$. Our next division gives $s = 2r + t$, where $t = 3 - 2\sqrt{2}$. Geometrically, if we lay off s along the diagonal, then r is the remainder $d - s$. Then draw a square of side r with part of the side of the original square being its diagonal. Note that if we now lay off r along the diagonal of that square, the remainder is t. In other words, r is the difference between the diagonal of a square of side s and s, while t is the difference between the diagonal of a square of side r and r. It follows that if one performs the next division in the process, we will get the same relationship. That is, $r = 2t + u$, where now u is the difference between the diagonal of a square of side t and t. Thus, this process will continue indefinitely and the ratio $d : s$ can be expressed as $(1, 2, 2, 2, \ldots)$.

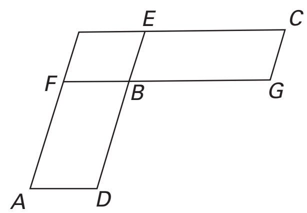
24. Since $a > b$, there is an integral multiple m of $a - b$ with $m(a - b) > c$. Let q be the first multiple of c that exceeds mb. Then $qc > mb \geq (q - 1)c$, or $qc - c \leq mb < qc$. Since $c < ma - mb$, it follows that $qc \leq mb + c < ma$. But also $qc > mb$. Thus we have a multiple (q) of c that is greater than a multiple (m) of b, while the same multiple (q) of c is not greater than the same multiple (m) of a. Thus by definition 7 of Book V, $c : b > c : a$.

25. Let $A : B = C : D = E : F$. We want to show that $A : B = (A + C + E) : (B + D + F)$. Take any equimultiples mA and $m(A + C + E)$ of the first and third and any equimultiples nB and $n(B + D + F)$ of the second and fourth. Since $m(A + C + E) = mA + mC + mE$, and since $n(B + D + F) = nB + nD + nF$, and since whenever $mA > nB$, we have $mC > nD$ and $mE > nF$, it follows that $mA > nB$ implies that $m(A + C + E) > n(B + D + F)$. Since a similar statement holds for equality and for "less than," the result follows from Eudoxus's definition. A modern proof would use the fact that $a_1 b_i = b_1 a_i$ for every i and then conclude that $a_1(b_1 + b_2 + \cdots + b_n) = b_1(a_1 + a_2 + \cdots + a_n)$.

26. Given that $a : b = c : d$, we want to show that $a : c = b : d$. So take any equimultiples ma, mb of a and b and also equimultiples nc, nd of c and d. Now $ma : mb = a : b = c : d = nc : nd$. Thus if $ma > nc$, then $mb > nd$; if $ma = nc$, then $mb = nd$; and if $ma < nc$, then $mb < nd$. Thus, by the definition of equal ratio, we have $a : c = b : d$.

27. Let $AB = 9$ and $BC = 5$. Draw a circle with AC as diameter and erect a perpendicular to AC at B, meeting the circle at E. Then BE is the desired length x.

28. Suppose the first of the equal and equiangular parallelograms has sides of length a and b while the second has sides of length c and d, each pair surrounding an angle equal to α. Since the area of a parallelogram is the product of the two sides with the sine of the included angle, we know that $ab \sin\alpha = cd \sin\alpha$. It follows that $ab = cd$ or that $a : c = d : b$, as desired. Conversely, if $a : c = d : b$ and the parallelograms are equiangular with angle α between each pair of given sides, then $ab = cd$ and $ab \sin\alpha = cd \sin\alpha$, so the parallelograms are equal. Euclid's proof is, of course, different from this modern one. Namely, if the two parallelograms are $P_1 = ADBF$ and $P_2 = BGCE$, with equal angles at B, Euclid places them so that FB and BG are in a straight line as are EB and BD. He then completes the third parallelogram $P_3 = FBEK$. Since $P_1 = P_2$, we have $P_1 : P_3 = P_2 : P_3$. But $P_1 : P_3 = DB : BE$, since BF is common, and $P_2 : P_3 = BG : BF$, since BE is common. Thus, $DB : BE = BG : BF$, the desired conclusion. The converse is proved by reversing the steps.



29 We want to prove that numbers to which the same number has the same ratio are equal. So suppose $a : b = a : c$. Therefore, a is the same multiple or the same part or the same parts of both b and c. That is, $a = mb$ and $a = mc$ for some rational number m. But this means that $b = \frac{1}{m}a$ and $c = \frac{1}{m}a$, thus $b = c$.

30. Suppose $a : b = f : g$ and suppose the numbers $c, d, \ldots, e$ are the numbers in continued proportion between a and b. Let $r, s, t, \ldots, u, v$ be the smallest numbers in the same ratio as $a, c, d, \ldots, e, b$. Then r, v are relatively prime and $r : v = a : b = f : g$. It follows that $f = mr$, $g = mv$ for some integer m and that the numbers $ms, mt, \ldots, mu$ are in the same ratio as the original set of numbers. Thus there are at least as many numbers in continued proportion between f and g as there were between a and b. Since the same argument works starting with f and g, it follows that there are exactly as many numbers in continued proportion between f and g as between a and b. Since there is no integer between n and $n+1$, it follows that there cannot be a mean proportional between any pair of numbers in the ratio $(n+1) : n$.

31. The number ab is the mean proportional between a^2 and b^2.

32. The numbers a^2b and ab^2 are the two mean proportionals between a^3 and b^3.

33. If a^2 measures b^2, then $b^2 = ma^2$ for some integer m. Since every prime number which divides b^2 must divide ma^2 and therefore must divide either m or a^2, it follows by counting primes that m must itself be a square. Thus $m = n^2$ and $b = na$, so a measures b. Conversely, if a measures b, then $b = na$ and $b^2 = n^2a^2$, so a^2 measures b^2.

34. Suppose m factors two different ways as a product of primes: $m = pqr \cdots s = p'q'r' \cdots s'$. Since p divides $pqr \cdots s$, it must also divide $p'q'r' \cdots s'$. By VII–30, p must divide one of the prime factors, say p'. But since both p and p' are prime, we must have $p = p'$. After canceling these two factors from their respective products, we can then repeat the argument to show that each prime factor on the left is equal to a prime factor on the right and conversely.

35. One standard modern proof is as follows. Assume there are only finitely many prime numbers $p_1, p_2, p_3, \ldots, p_n$. Let $N = p_1p_2p_3 \cdots p_n + 1$. There are then two possibilities. Either N is prime or N is divisible by a prime other than the given ones, since division by any of those leaves remainder 1. Both cases contradict the original hypothesis, which therefore cannot be true.

36. We keep adding powers of 2 until we get a prime. After $1+2+4+8+16+32+64 = 127$, the next sums are 255, 511, 1023, 2047, 4095, and 8191. The first five of these are not prime (note that $2047 = 23 \times 89$). But 8191 is prime (check by dividing by all primes less than $\sqrt{8191}$). So the next perfect number is $8191 \times 4096 = 33{,}550{,}336$.

37. Since BC is the side of a decagon, triangle EBC is a 36-72-72 triangle. Thus $\angle ECD = 108^\circ$. Since CD, the side of a hexagon, is equal to the radius CE, it follows that triangle ECD is an isosceles triangle with base angles equal to 36°. Thus triangle EBD is a 36-72-72 triangle and is similar to triangle EBC. Therefore $BD : EC = EB : BC$ or $BD : CD = CD : BC$ and the point C divides the line segment BD in extreme and mean ratio.

38. By Exercise 37, if we set d to be the length of the side of a decagon, we have $(1+d) : 1 = 1 : d$ or $d^2 + d - 1 = 0$. It follows that $d = \frac{\sqrt{5}-1}{2}$. The length p of the side of a pentagon is (p. 91) $p = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}$. It is then straightforward to show that $p^2 = 1^2 + d^2$ as asserted.

39. The edge length of the icosahedron in a sphere of diameter 1 is $e_i = \frac{1}{10}\sqrt{50 - 10\sqrt{5}} = 0.52573111$. The circumradius of the equilateral triangle with this edge length is $r = \frac{e_i}{2\sin 60^\circ} = 0.303531$. The edge length of the dodecahedron in the same sphere is $e_d = \frac{1}{6}(\sqrt{15} - \sqrt{3}) = 0.35682209$. The circumradius of the pentagon with this edge length is $r = \frac{e_d}{2\sin 36^\circ} = 0.303531$, the same value as in the icosahedron.

40. We begin with a rectangle of sides x and y. We then lay off y along x, with the remainder being $x - y = 7$. Divide the rectangle with sides 7 and y in half and move one half to the bottom. We then add the square of side $\frac{7}{2}$ to get a square of side $y + \frac{7}{2} = x - \frac{7}{2}$ with area $18 + (\frac{7}{2})^2 = \frac{121}{4}$. Thus $y = \frac{11}{2} - \frac{7}{2} = 2$ and $x = \frac{11}{2} + \frac{7}{2} = 9$.

41. First, to solve the two equations, we set $x = a/y$ and substitute into $y^2 - \alpha x^2 = b$. After multiplying by y^2, we get the fourth degree equation $y^4 - \alpha a^2 = by^2$. This is quadratic in y^2, so we can solve to get

$$y^2 = \frac{b + \sqrt{b^2 + 4\alpha a^2}}{2}.$$

(Note that we only use the plus sign, since y^2 must be positive.) Then

$$y = \pm\sqrt{\frac{b + \sqrt{b^2 + 4\alpha a^2}}{2}}.$$

We can then solve for x:

$$x = \pm\sqrt{\frac{\sqrt{b^2 + 4\alpha a^2} - b}{2\alpha}}.$$

(Of course, since a is assumed positive, we take the positive solution for x when we have the positive one for y, as well as the negative solution for x with the negative one for y.) The asymptotes of the hyperbola $xy = a$ are the lines $x = 0$ and $y = 0$, and these are the axes of the hyperbola $y^2 - \alpha x^2 = b$. Thus the asymptotes of one hyperbola are the axes of the other.

CHAPTER 4

1. If x is the distance to the 14 kg end, then $10(10 - x) = 14x$, $100 = 24x$, and $x = 4\frac{1}{6}$ m from the 14 kg end.
2. Since $8 \cdot 10 = 80$ and $12 \cdot 8 = 96$, the lever inclines toward the 12 kg weight.
3. Since a weight W of gold displaces a volume V_1 of fluid, a weight w_1 of gold will displace $\frac{w_1}{W} \cdot V_1$ of fluid. Similarly, a weight w_2 of silver displaces $\frac{w_2}{W} \cdot V_2$ of fluid. Thus the wreath, of total weight W, displaces the sum of these two amounts of fluid. That is, $V = \frac{w_1}{W} \cdot V_1 + \frac{w_2}{W} \cdot V_2$. Since $W = w_1 + w_2$, it follows that $(V - V_1)w_1 = (V_2 - V)w_2$, and the desired result follows.
4. Lemma 1: $DA/DC = OA/OC$ by *Elements* VI–3. Therefore $DA/OA = DC/OC = (DC+DA)/(OC+OA) = AC/(CO+OA)$. Also, $DO^2 = OA^2+DA^2$ by the Pythagorean Theorem.

   Lemma 2: $AD/DB = BD/DE = AC/CE = AB/BE = (AB+AC)/(CE+BE) = (AB+AC)/BC$. Therefore, $AD^2/BD^2 = (AB+AC)^2/BC^2$. But $AD^2 = AB^2 - BD^2$. So $(AB^2 - BD^2)/BD^2 = (AB+AC)^2/BC^2$ and $AB^2/BD^2 = 1 + (AB+AC)^2/BC^2$.
5. Set $r = 1$, t_i and u_i as in the text, and P_i the perimeter of the ith circumscribed polygon. Then the first ten iterations of the algorithm give the following:

$t_1 = 0.577350269$	$u_1 = 1.154700538$	$P_1 = 3.464101615$
$t_2 = 0.267949192$	$u_2 = 1.03527618$	$P_2 = 3.21539031$
$t_3 = 0.131652497$	$u_3 = 1.008628961$	$P_3 = 3.159659943$
$t_4 = 0.065543462$	$u_4 = 1.002145671$	$P_4 = 3.146086215$
$t_5 = 0.03273661$	$u_5 = 1.0005357$	$P_5 = 3.1427146$
$t_6 = 0.016363922$	$u_6 = 1.00013388$	$P_6 = 3.141873049$
$t_7 = 0.0081814134$	$u_7 = 1.000033467$	$P_7 = 3.141662746$
$t_8 = 0.004090638249$	$u_8 = 1.000008367$	$P_8 = 3.141610175$
$t_9 = 0.002045310568$	$u_9 = 1.000002092$	$P_9 = 3.141597032$
$t_{10} = 0.001022654214$	$u_{10} = 1.000000523$	$P_{10} = 3.141593746$

6. Let d be the diameter of the circle, t_i the length of one side of the regular inscribed polygon of $3 \cdot 2^i$ sides, and u_i the length of the other leg of the right triangle formed from the diameter and the side of the polygon. Then

$$\frac{{t_{i+1}}^2}{d^2} = \frac{t_i^2}{t_i^2 + (d+u_i)^2}$$

or

$$t_{i+1} = \frac{dt_i}{\sqrt{t_i^2 + (d+u_i)^2}} \qquad u_{i+1} = \sqrt{d^2 - {t_{i+1}}^2}.$$